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Partitions of R^3 in unit circles

and the Axiom of Choice

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Winter school in Abstract Analysis: section Set Theory and Topology

Good morning!

Goal:

Present a model of

ZF + $\exists$ partition of $\mathbb{R}^3$ in unit circles
+ no well-order of the reals.

— This is joint work (in progress) with Prof. Ralf Schindler —

But why?

Context

- My interest: Paradoxical sets and the relation with AC

$$\text{AC} \Leftrightarrow \text{Well-order of } \mathbb{R}$$

Well-order of $\mathbb{R}$ $\overset{\Rightarrow}{\underset{?}{\Leftarrow}}$

$\exists$ Hamel basis

$\exists$ Mazurkiewicz set

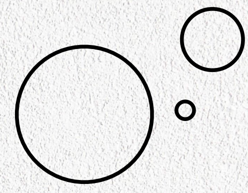
$\exists$ Partition of $\mathbb{R}^3$ in unit circles

...

Context

- Mathematical context:

(1) ZF ⊢ $\mathbb{R}^3$ can be partitioned in circles →



R^3 IS THE UNION OF DISJOINT CIRCLES (1983)

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(2) ZFC ⊢ $\mathbb{R}^3$ can be partitioned in unit circles

Covering a sphere with congruent great-circle arcs

BY J. H. CONWAY AND H. T. CROFT (1964)
Gonville and Caius College, and Peterhouse, Cambridge

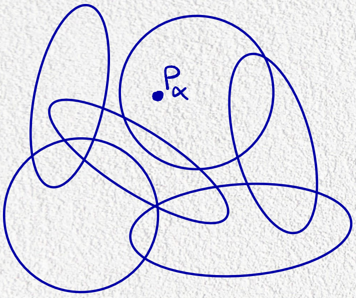
A.B. Kharazishvili. Partition of three-dimensional space with congruent circles. Bull. Acad. Sci. Georgian SSR, 119 (1985), 57-60 (in Russian).

Context

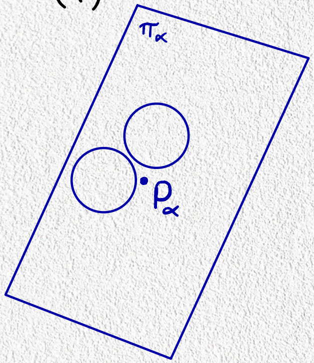
Theorem (ZFC) (Conway, Croft / Kharazishvili)
$\mathbb{R}^3$ can be partitioned in unit circles

Proof: $\mathbb{R}^3 = \{p_\alpha\}_{\alpha < c}$

(0)



(1)



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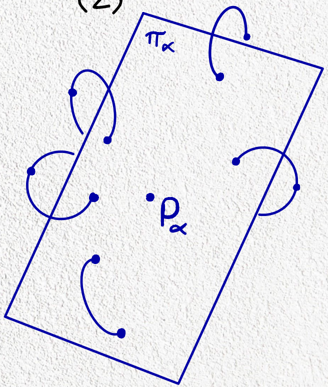


Definitions

- PUC

P	U	C
a	n	i
r	i	r
t	t	c
i		l
t		e
i		s
o		
n		

- partial PUC

Observation: The proof shows that any partial PUC of cardinality $<\mathfrak{c}$ can be extended to a (complete) PUC.

Question: Can we always extend a partial PUC to a (complete) PUC?

Definitions

Sometimes there is not enough "space" to extend a partial PUC.

Let's create more space!

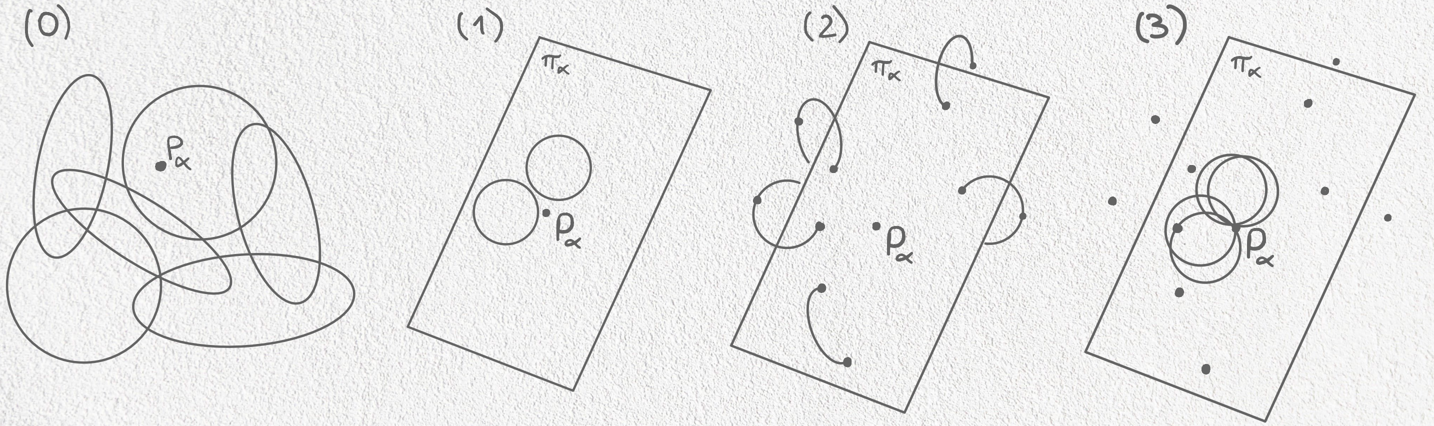
Fact: Let $V \models ZFC$ and $V[r]$ be a generic extension obtained by adding one Cohen real. Then the transcendence degree of $\mathbb{R}^{V[r]}$ over $\mathbb{R}^{V}$ is $\mathfrak{c}$.

Leap of faith

Lemma 1 (Extendability)

Let V be a ZFC model and $p \in V$ such that $V \models$ "p is a partial PUC". Let r be a Cohen real over V. Then, there is $q \in V[r]$ s.t. $V[r] \models$ "$q \supseteq p \wedge q$ is a PUC"

Proof: $\mathbb{R}^3 = \{p_\alpha\}_{\alpha < \mathfrak{c}}$

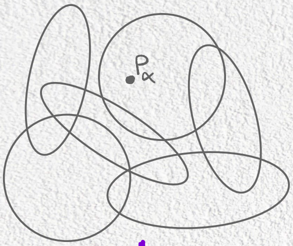


Leap of faith

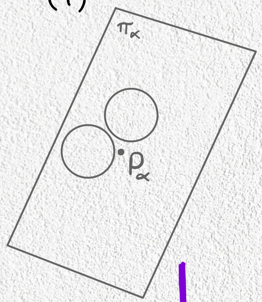
Lemma 1 (Extendability)
Let V be a ZFC model and $p \in V$ such that $V \models$ "p is a partial PUC". Let r be a Cohen real over V. Then, there is $q \in V[r]$ s.t. $V[r] \models$ "$q \geq p \wedge q$ is a PUC"

Proof: $\mathbb{R}^3 \setminus \bigcup p = \{p_\alpha\}_{\alpha < \mathfrak{c}}$

(0)



(1)



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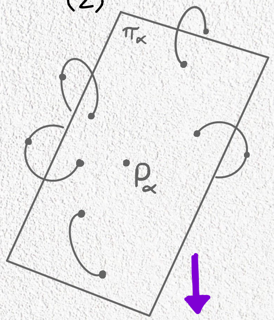

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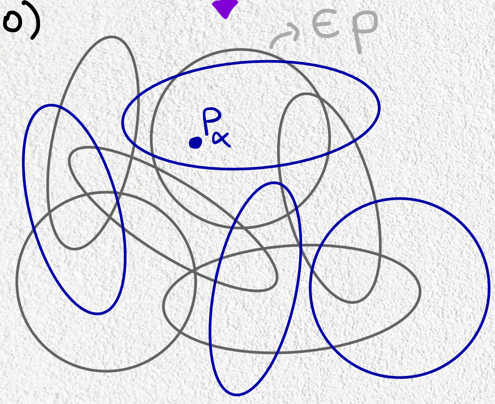
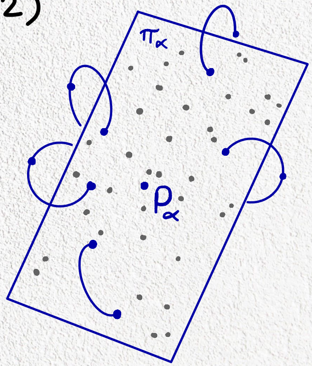


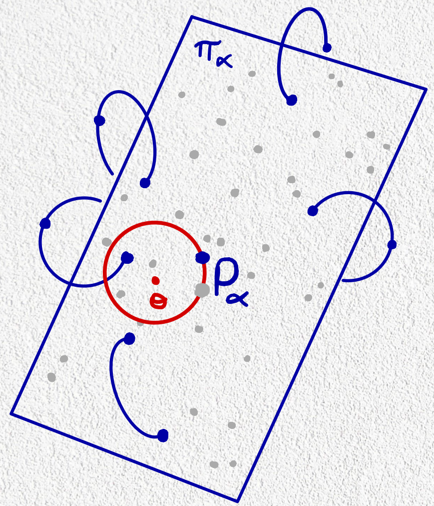
(1) How do we pick π_α?

(2)



Leap of faith

Proof: $\mathbb{R}^3 \setminus \bigcup \mathcal{P} = \{p_\alpha\}_{\alpha < \mathfrak{c}}$



Pick θ_α "algebraically independent" from …

$\square$

But why?

But why?

Goal: Present a model of

$$\text{ZF} + \exists \text{ partition of } \mathbb{R}^3 \text{ in unit circles} + \text{no well-order of the reals.}$$

We follow the structure of the construction of a model of ZF with a Hamel basis of $\mathbb{R}$ but no well-order of the reals done in:

A MODEL WITH EVERYTHING EXCEPT FOR A WELL-ORDERING OF THE REALS

JÖRG BRENDLE, FABIANA CASTIBLANCO, RALF SCHINDLER, LIUZHEN WU, AND LIANG YU

as we did also for the construction of a model of ZF without a well-order of the reals in which there is a Mazurkiewicz set.

"The model"

$$L \xrightarrow{\mathbb{C}_{\omega_1}} L[g] \xrightarrow{\mathbb{P}_H} L[g,h]$$

$$\boxed{L(\mathbb{R}, b)^{L[g,h]}}, \quad \text{where } b = \bigcup h$$

- $p \in \mathbb{P}_H$ iff $\exists x \in \mathbb{R}$ such that:
  - $p \in L[x]$
  - $L[x] \models p$ is a Hamel basis
- $p \leq_{\mathbb{P}_H} q$ iff $p \supseteq q$.

"The model"

$$\begin{tikzcd} L \arrow[r, squiggly, "\mathbb{C}_{\omega_1}"] & L[g] \arrow[r, squiggly, "\mathbb{P}_{\times}"] & L[g,h] \arrow[d, squiggly] \\ & & \end{tikzcd}$$

$$W = \boxed{L(\mathbb{R}, b)^{L[g,h]}}, \quad \text{where } b = \bigcup h$$

- $p \in \mathbb{P}_{\times}$ iff $\exists x \in \mathbb{R}$ such that:
  - $p \in L[x]$
  - $L[x] \models p$ is a ~~Hamel basis~~ PUC
- $p \leq_{\mathbb{P}_{\times}} q$ iff $p \supseteq q$ AND...

Why does this model work?

- Forcing with $\mathbb{P}$ adds a PUC,
- We did not add reals,
- $W \models$ there is no well-order of the reals.

How the proof looks like $\quad (m = b)$

I. $L[g,h] \models$ "$\varphi(\cdot,\cdot,\vec{x},\vec{\alpha},m)$ defines a well-ordering of ${}^{\omega}2$",

II. $p \Vdash^{\mathbb{P}}_{L[g\restriction\alpha][g\restriction[\alpha,\omega_1)]}$ "$\varphi(\cdot,\cdot,\check{\vec{x}},\check{\vec{\alpha}},\dot{m})$ defines a well-ordering of ${}^{\omega}\check{2}$"

III. $\mathbb{1} \Vdash^{\mathbb{C}(\omega_1)}_{L[g\restriction\alpha]} \check{p} \Vdash^{\mathbb{P}}_{L[g\restriction\alpha][\dot{g}]}$ "$\varphi(\cdot,\cdot,\check{\check{\vec{x}}},\check{\check{\vec{\alpha}}},\dot{m})$ defines a well-ordering of ${}^{\omega}2$".

IV. $L[g\restriction\alpha, g^*][h^*] \models$ "$\varphi(\cdot,\cdot,\vec{x},\vec{\alpha},m^*)$ defines a well-ordering of ${}^{\omega}2$",

Since $\mathbb{R}\cap L[g\restriction\alpha, g^*][h^*] = \mathbb{R}\cap L[g\restriction\alpha, g^*] \neq \mathbb{R}\cap L[g\restriction\alpha, g\restriction[\alpha,\omega_1)] = \mathbb{R}\cap L[g,h]$

V
i) $L[g,h] \models$ "the n^{th} digit of the η^{th} element of ${}^{\omega}2$ given by $\varphi(\cdot,\cdot,\vec{x},\vec{\alpha},m)$ is i"
ii) $L[g\restriction\alpha, g^*][h^*] \models$ "the n^{th} digit of the η^{th} element of ${}^{\omega}2$ given by $\varphi(\cdot,\cdot,\vec{x},\vec{\alpha},m^*)$ is $1-i$"

VI
i)* $p_0 \Vdash^{\mathbb{P}}_{L[g]}$ "the $\check{n}^{th}$ digit of the $\check{\eta}^{th}$ element of ${}^{\omega}2$ given by $\varphi(\cdot,\cdot,\check{\vec{x}},\check{\vec{\alpha}},\dot{m})$ is $\check{i}$"
ii)* $p_1 \Vdash^{\mathbb{P}}_{L[g\restriction\alpha, g^*]}$ "the $\check{n}^{th}$ digit of the $\check{\eta}^{th}$ element of ${}^{\omega}2$ given by $\varphi(\cdot,\cdot,\check{\vec{x}},\check{\vec{\alpha}},\dot{m})$ is $1-\check{i}$"

VII
$\mathbb{1} \Vdash^{\mathbb{C}(\omega_1)^2}_{L[g\restriction\beta]} \check{p}_0 \Vdash^{\mathbb{P}}_{L[g\restriction\beta][\dot{g}]}$ "the $\check{\check{n}}^{th}$ digit of the $\check{\check{\eta}}^{th}$ element of ${}^{\omega}2$ given by $\varphi(\cdot,\cdot,\check{\check{\vec{x}}},\check{\check{\vec{\alpha}}},\dot{m})$ is $\check{\check{i}}$"

$\mathbb{1} \Vdash^{\mathbb{C}(\omega_1)^2}_{L[g\restriction\alpha, g^*\restriction\beta]} \check{p}_1 \Vdash^{\mathbb{P}}_{L[g\restriction\alpha, g^*\restriction\beta][\dot{g}]}$ "the $\check{\check{n}}^{th}$ digit of the $\check{\check{\eta}}^{th}$ element of ${}^{\omega}2$ given by $\varphi(\cdot,\cdot,\check{\check{\vec{x}}},\check{\check{\vec{\alpha}}},\dot{m})$ is $1-\check{\check{i}}$"

- We have two conditions that force incompatible statements.

Lemma 3 (Amalgamation)

Let x, y, z mutually generic Cohen reals and let p, q_1, q_2 be such that

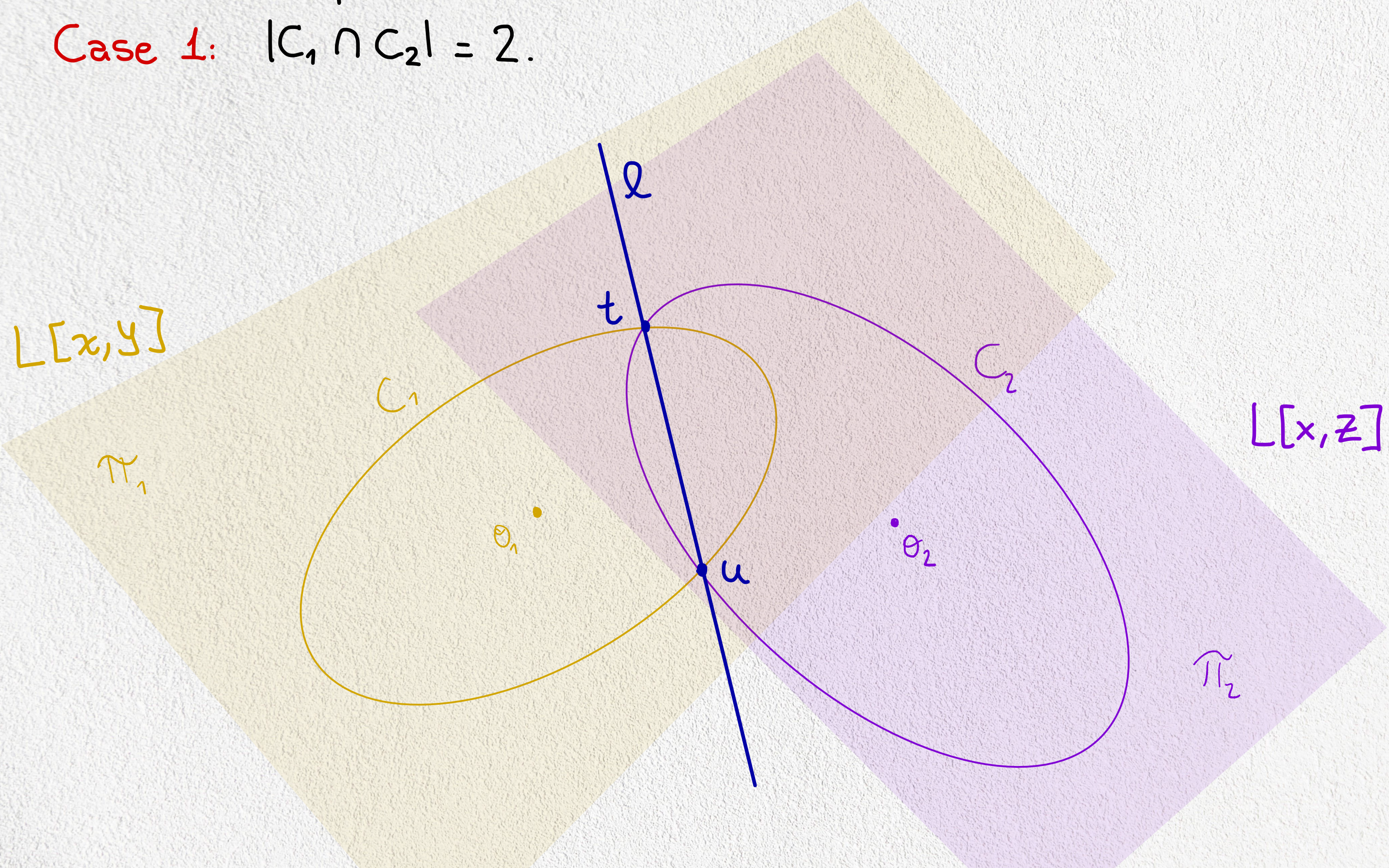
$$\begin{cases} L[x] \models p \text{ is a PUC} \\ L[x,y] \models q_1 \text{ is a PUC} \\ L[x,z] \models q_2 \text{ is a PUC} \end{cases}$$

and $q_1, q_2 \leq_{\mathbb{P}} p$.

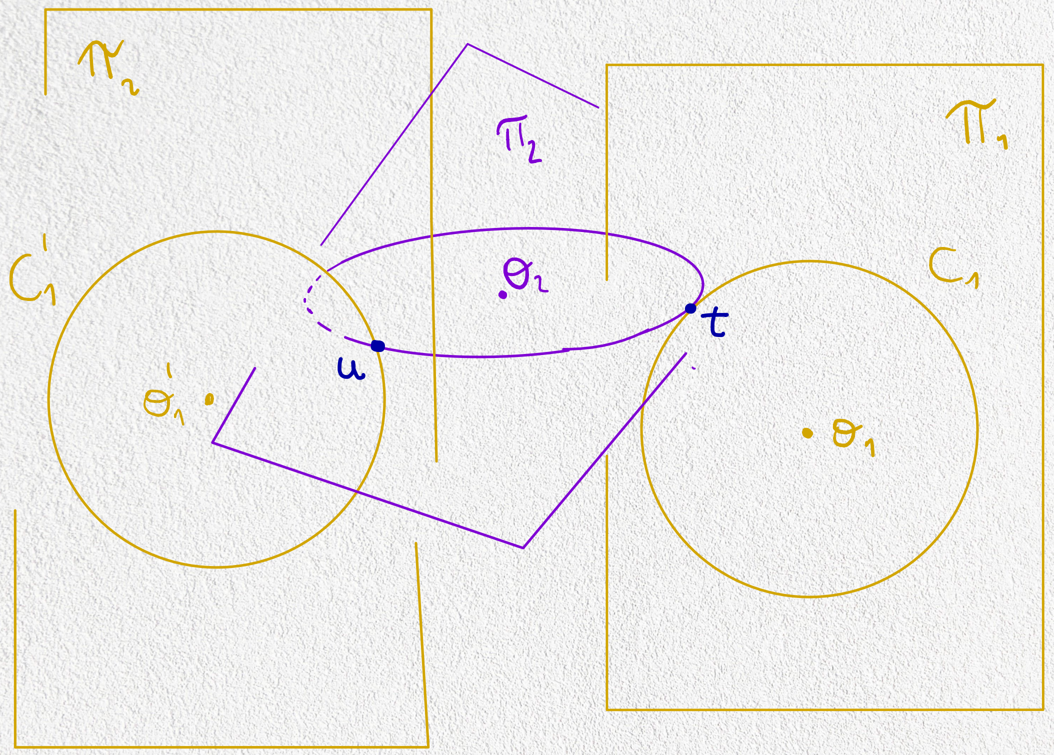
Then $L[x,y,z] \models q_1 \cup q_2$ is a partial PUC.

Proof: Suppose not. $C_1 \in q_1$, $C_2 \in q_2$, $C_1 \cap C_2 \neq \emptyset$.

Case 1: $|C_1 \cap C_2| = 2$.

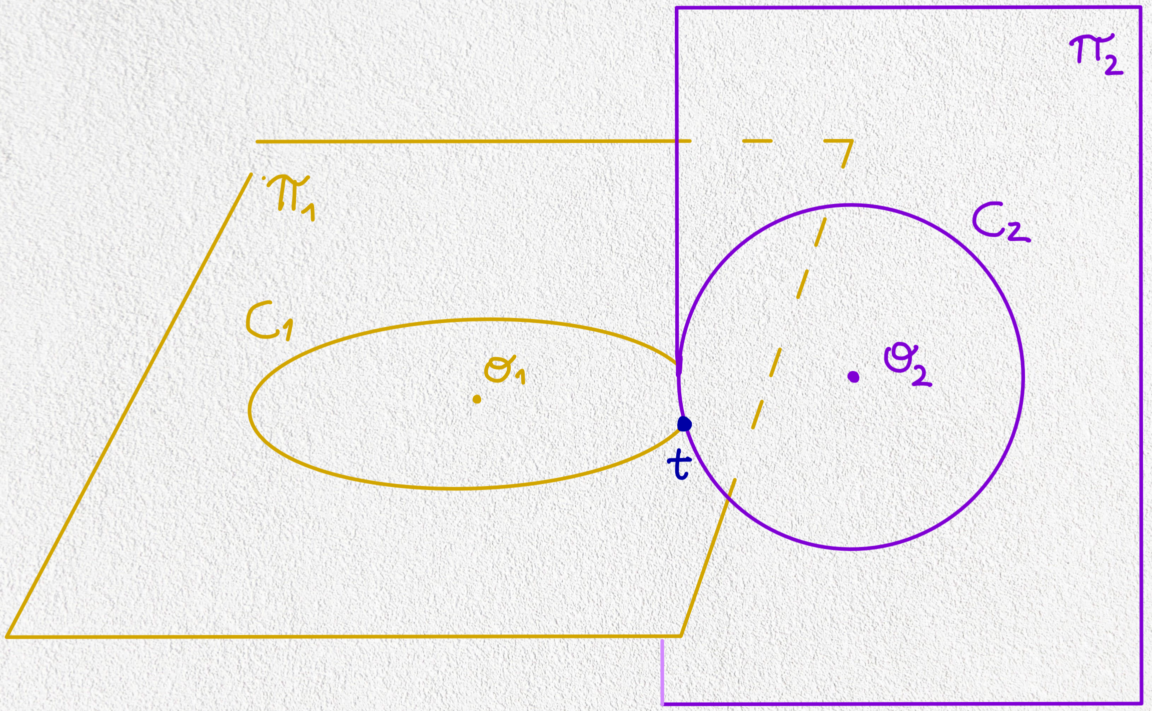


Actually ...



... this (↑) can not happen either.

Case 2: $|C_1 \cap C_2| = 1$ and C_2 is the only circle "from" $L[x,z]$ that intersects C_1.



Thank you!

References

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